Bertrand spacetimes as Kepler/oscillator potentials

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Abstract

Perlick's classification of $(3+1)$-dimensional spherically symmetric and static spacetimes $(\mathcal{M}, \eta = -\frac{1}{V}\, \mathrm{d}t^2 + g)$ for which the classical Bertrand theorem holds [Perlick V 1992 *Class. Quantum Grav.* **9** 1009] is revisited. For any Bertrand spacetime $(\mathcal{M}, \eta)$ the term $V(r)$ is proven to be either the intrinsic Kepler–Coulomb or the harmonic oscillator potential on its associated Riemannian 3-manifold (M, g). Among the latter 3-spaces (M, g) we explicitly identify the three classical Riemannian spaces of constant curvature, a generalization of a Darboux space and the Iwai–Katayama spaces generalizing the MIC–Kepler and Taub-NUT problems. The key dynamical role played by the Kepler and oscillator potentials in Euclidean space is thus extended to a wide class of 3-dimensional curved spaces.

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1 Introduction

It is a well known, but nonetheless quite remarkable fact that all the bounded trajectories in Euclidean space associated with the Kepler–Coulomb (KC) potential $V(r) = \alpha/r + \beta$ are periodic, so that they yield close orbits, i.e. ellipses. This is in striking contrast e.g. with the gravitational 3-body problem, which is known to possess extremely complicated bounded orbits. Similarly, the orbits of the harmonic oscillator potential $V(r) = \omega^2 r^2 + \delta$ are all closed. The content of the celebrated theorem of Bertrand [1] is that, under certain technical conditions, the KC and harmonic oscillator potentials are the only spherically symmetric ones for which all the bounded trajectories of the Hamiltonian $H(\mathbf{p}, \mathbf{q}) = \mathbf{p}^2 + V(|\mathbf{q}|)$ are periodic. The proof of Bertrand's theorem is not difficult, and has actually been included (with various levels of rigor) in several textbooks. Nevertheless, this classical result provides a wealth of subtle connections with different areas of Physics, and still attracts considerable attention, mainly motivated by its deep connection with accidental degeneracy and with superintegrability in quantum and classical mechanics [2, 3, 4, 5].

Some years ago, in a remarkable paper, Perlick [6] showed that Bertrand's theorem also arises naturally in General Relativity. Indeed, let us consider a spherically symmetric static spacetime $(\mathcal{M}, \eta)$ which is the Lorentzian warped product of the line by a Riemannian 3-dimensional (3D) manifold (M, g):

$$\eta = -\frac{1}{V}\,\mathrm{d}t^2 + g\,. \tag{1.1}$$

Of course, the warping function $V \in C^\infty(M)$ must be strictly positive. It can be easily seen that the projection of each timelike geodesic on a constant time leaf $M_0 = M \times \{t_0\}$ is in fact a trajectory of the natural Hamiltonian $H(\mathbf{p}, \mathbf{q}) = \|\mathbf{p}\|^2_{T^*M_0} + V(\mathbf{q})$. This relationship between an autonomous natural Hamiltonian flow on a 3D manifold with lower bounded potential and the timelike geodesics of a Lorentzian causal manifold can be understood as the Lorentzian analog of the introduction of the Jacobi metric in classical mechanics [7]. Consequently, by a *trajectory* in $(\mathcal{M}, \eta)$ we shall mean the projection of a timelike geodesic onto M_0.

Following Perlick, we shall say that a domain $\mathcal{B}$ of a smooth $(3+1)$D Lorentzian spacetime $(\mathcal{M}, \eta)$ (possibly the whole space) is *a Bertrand spacetime* provided that:

1. $(\mathcal{M}, \eta)$ is spherically symmetric and static, the domain $\mathcal{B}$ (possibly minus one or two points) is diffeomorphic to a product manifold $]r_1, r_2[\times \mathbb{S}^2 \times \mathbb{R}$ and the metric η in $\mathcal{B}$ takes the form

$$\eta = -\frac{1}{V(r)}\,\mathrm{d}t^2 + g(r)^2 \mathrm{d}r^2 + r^2\big(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\big). \tag{1.2}$$

2. There is a circular ($r =$ constant) trajectory passing through every point of $\mathcal{B}$.

3. Circular trajectories are stable under small perturbations of the initial conditions.

Perlick [6] posed and solved the problem of classifying all Bertrand spacetimes, finding that the only possibilities are *three* multiparametric families of spacetimes, called hereafter of type I and $\mathrm{II}_\pm$, which are explicitly given by

$$\text{Type I}: \qquad \mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{G+\sqrt{r^{-2}+K}} + \frac{\mathrm{d}r^2}{\beta^2\,(1+Kr^2)} + r^2(\mathrm{d}\vartheta^2+\sin^2\vartheta\,\mathrm{d}\varphi^2) \tag{1.3}$$

$$\text{Type II}_\pm: \qquad \mathrm{d}s^2 = -\frac{\mathrm{d}t^2}{G\mp r^2\left(1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4}\right)^{-1}} \tag{1.4}$$
$$+\frac{2\left(1-Dr^2\pm\sqrt{(1-Dr^2)^2-Kr^4}\right)}{\beta^2\left((1-Dr^2)^2-Kr^4\right)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2+\sin^2\vartheta\,\mathrm{d}\varphi^2)$$

where D, G and K are real constants, β is a positive rational number, r is a radial coordinate restricted to certain interval $]r_1, r_2[$, (ϑ,φ) are spherical coordinates on $\mathbb{S}^2$ and $t\in\mathbb{R}$.

Bertrand spacetimes are interesting in their own right because of their definition in terms of closed orbits. From the point of view of manifold theory, closed geodesics have long played a preponderant role in Riemannian geometry [8]. A somewhat similar question, that of characterizing all Riemannian manifolds whose geodesics are all closed, is still wide open [9, 10]. Periodic trajectories are also central in the theory of dynamical systems, where a generic dynamical system with a compact invariant set does have many closed orbits [11]. Obstructions do arise, of course, when considering regular vector fields whose orbits are all circles.

The aim of this paper is to prove that, given any $(3+1)$D Bertrand spacetime with Lorentzian metric η (1.1), the term $V(r)$ can *always* be interpreted as either the intrinsic KC or the harmonic oscillator potential on the corresponding 3D Riemannian Bertrand space (M,g). Therefore, we do not only have a result on closed trajectories of spherically symmetric spaces that extends Bertrand's theorem to 3D curved manifolds, but we also recover the key role played by the corresponding KC and harmonic oscillator potentials of these 3D spaces.

In this respect, we stress that Perlick, in a purely relativistic setting, proved that KC-like and oscillator-like $(3+1)$D metrics can only be identified provided one considers, in this order, Bertrand spacetimes given by the values $\beta=1$ and $\beta=2$ [6]. Our results show that a proper (non-relativistic) Kepler/oscillator potential can always be established for any 3D Bertrand space (M,g), independently of the value of β, and such Kepler/oscillator potentials give all the possible $V(r)$ terms in Perlick's classification of Bertrand spacetimes.

Furthermore, we discuss explicitly some particular Bertrand spaces of interest, such as the three classical Riemannian spaces of constant curvature [12] (this case was briefly pointed out by Perlick [6]), a Darboux space with non-constant curvature [13] as well as the spaces introduced by Iwai and Katayama [14, 15] as a generalization of the MIC–Kepler [16, 17, 18] and Taub–NUT metrics [19]. We recall that the Euclidean Taub–NUT metric has attracted considerable attention in the physics community (see, for instance, [20, 21, 22, 23, 24, 25] and references therein) since, in particular, the

relative motion of two monopoles in this metric is asymptotically described by its geodesics [26]. Moreover, the reduction of the Euclidean Taub–NUT space by its U(1)-symmetry, as performed in the analysis of its geodesics, essentially gives rise to a specific Iwai–Katayama space [14]. In this way, all of these different systems are unified within our approach in a common Bertrand space framework.

The article is organized as follows. In section 2 we present the metric structure of the 3D Bertrand spaces by considering them within a particular class of spherically symmetric Riemannian spaces. In section 3 we identify some physically relevant examples of Bertrand spaces. The construction (and interpretation) of the intrinsic KC and harmonic oscillator potentials on generic Bertrand spaces is addressed in section 4. In section 5, we explicitly study the KC and oscillator potentials for the metrics described in section 3. Some comments concerning the generalization of the results here presented and their connection with superintegrability properties of the associated Hamiltonian flows are drawn in the last section.

2 Bertrand spaces

Each of the (Lorentzian) Perlick metrics [6] defines the kinetic energy on the $(3+1)$D manifold $\mathcal{M}$. Alternatively, this construction provides a natural Hamiltonian on the 3D spherically symmetric space (M, g) with coordinates (r, ϑ, φ) and with a central potential $V(r)$ given by the denominator of the term in $\mathrm{d}t^2$ (1.2). Therefore, according to (1.3) and (1.4), hereafter we shall consider the Bertrand spaces defined by

$$\text{Type I}: \qquad \mathrm{d}s^2 = \frac{\mathrm{d}r^2}{\beta^2\,(1+Kr^2)} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \tag{2.1}$$

$$\text{Type II}_\pm: \qquad \mathrm{d}s^2 = \frac{2\left(1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}\right)}{\beta^2\left((1-Dr^2)^2 - Kr^4\right)}\,\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2). \tag{2.2}$$

Consequently, Bertrand spaces are included within the large family of 3D spherically symmetric spaces. Their corresponding general metric is usually written in terms of a 'conformal' factor depending on a radial coordinate, say $\tilde r$ (different from r), and sometimes in terms of its logarithm ($\rho = \ln \tilde r$). Hence in order to be able to relate our results with others given in the literature we shall consider the following generic metric in a 3D Riemannian manifold M written in different coordinate systems:

$$\begin{aligned} \mathrm{d}s^2 &= f(|\mathbf{q}|)^2\,\mathrm{d}\mathbf{q}^2 = f(\tilde r)^2(\mathrm{d}\tilde r^2 + \tilde r^2\mathrm{d}\Omega^2) \\ &= F(\rho)^2\,(\mathrm{d}\rho^2 + \mathrm{d}\Omega^2) = g(r)^2\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \end{aligned} \tag{2.3}$$

where $|\mathbf{q}|^2 = \mathbf{q}^2 = \sum_{i=1}^3 q_i^2$ and $\mathrm{d}\mathbf{q}^2 = \sum_{i=1}^3 \mathrm{d}q_i^2$; $f(|\mathbf{q}|) \equiv f(\tilde r)$, $F(\rho)$ and $g(r)$ are smooth functions; and $\mathrm{d}\Omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2$ is the standard metric on the unit 2D sphere $\mathbb{S}^2$. The relations between the coordinate systems, (q_1, q_2, q_3), $(\tilde r, \vartheta, \varphi)$, $(\rho, \vartheta, \varphi)$

and (r, ϑ, φ) are given by

$$\begin{aligned} q_1 &= \tilde{r} \cos\vartheta \qquad q_2 = \tilde{r} \sin\vartheta \cos\varphi \qquad q_3 = \tilde{r} \sin\vartheta \sin\varphi \\ |\mathbf{q}| &= \tilde{r} \qquad \rho = \ln \tilde{r} \qquad F(\rho) = \tilde{r} f(\tilde{r}) \qquad r = F(\rho) \qquad \mathrm{d}\rho = r^{-1} g(r) \mathrm{d}r. \end{aligned} \tag{2.4}$$

The scalar curvature R of the metric (2.3) is always determined by the 'radial' (but not necessarily geodesic!) coordinate $\tilde{r} \leftrightarrow \rho \leftrightarrow r$. This curvature is generically non-constant and turns out to be

$$\begin{aligned} R &= 2\left(\frac{f'(\tilde{r})^2 - 2f(\tilde{r})\left(f''(\tilde{r}) + 2\tilde{r}^{-1} f'(\tilde{r})\right)}{f(\tilde{r})^4}\right) \\ &= 2\left(\frac{F'(\rho)^2 + F(\rho)^2 - 2F(\rho)F''(\rho)}{F(\rho)^4}\right) = 2\left(\frac{g(r)^2 + 2rg'(r)/g(r) - 1}{r^2 g(r)^2}\right) \end{aligned} \tag{2.5}$$

where a prime denotes derivative with respect to the corresponding argument, i.e. $f' = \mathrm{d}f/\mathrm{d}\tilde{r}$ and so on.

2.1 Spaces of type I

If we now apply the above results to the Bertrand metrics of type I (2.1) we find that

$$g(r) = \frac{1}{\beta\sqrt{1 + Kr^2}} \qquad \mathrm{d}\rho = \frac{\mathrm{d}r}{\beta r\sqrt{1 + Kr^2}} \tag{2.6}$$

which yield

$$\rho = \frac{1}{\beta} \ln\left(\frac{r}{1 + \sqrt{1 + Kr^2}}\right) \quad \text{or} \quad \tilde{r} = \left(\frac{r}{1 + \sqrt{1 + Kr^2}}\right)^{1/\beta}. \tag{2.7}$$

Thus the Perlick radial coordinate r is given in terms of the 'conformal' ones by

$$r = \frac{2}{\tilde{r}^{-\beta} - K\tilde{r}^{\beta}} = \frac{2}{\mathrm{e}^{-\beta\rho} - K\mathrm{e}^{\beta\rho}}. \tag{2.8}$$

Hence the metric (2.1) can also be written as

$$\mathrm{d}s^2 = \frac{4}{(\tilde{r}^{-\beta} - K\tilde{r}^{\beta})^2}\left(\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2\right) = \frac{4}{(\mathrm{e}^{-\beta\rho} - K\mathrm{e}^{\beta\rho})^2}\left(\mathrm{d}\rho^2 + \mathrm{d}\Omega^2\right) \tag{2.9}$$

and its scalar curvature reads

$$\begin{aligned} R &= -\frac{1}{2}\left\{(\beta^2 - 1)(K^2\tilde{r}^{2\beta} + \tilde{r}^{-2\beta}) + 2K(1 + 5\beta^2)\right\} \\ &= -\frac{1}{2}\left\{(\beta^2 - 1)(K^2\mathrm{e}^{2\beta\rho} + \mathrm{e}^{-2\beta\rho}) + 2K(1 + 5\beta^2)\right\} \\ &= -\frac{2}{r^2}\left\{3\beta^2 K r^2 + \beta^2 - 1\right\}. \end{aligned} \tag{2.10}$$

2.2 Spaces of type $\mathrm{II}_\pm$

As far as the Bertrand type $\mathrm{II}_\pm$ is concerned, the function $g(r)$ is given by

$$g(r) = \frac{\sqrt{2}}{\beta} \left(\frac{1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}}{(1-Dr^2)^2 - Kr^4} \right)^{1/2} . \tag{2.11}$$

The explicit form of $\rho(r)$ can be obtained by taking into account that:

$$\rho(r) = \int^r \frac{g(r)}{r} \mathrm{d}r = \int^r r \, \frac{g(r)}{r^2} \mathrm{d}r = r \int^r \frac{g(r)}{r^2} \mathrm{d}r - \int^r \left(\int^r \frac{g(r)}{r^2} \mathrm{d}r \right) \mathrm{d}r$$

which yields

$$\rho(r) = rU(r) - \int^r U(r) \mathrm{d}r \tag{2.12}$$

where

$$U(r) = \int^r \frac{g(r)}{r^2} \mathrm{d}r = \mp \frac{\sqrt{2}}{\beta r} \left(1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4} \right)^{1/2} . \tag{2.13}$$

The integral $\int^r U(r)\mathrm{d}r$, although it can be explicitly written, is quite involved, so that we omit it and present the results for this family of metrics just in terms of r.

The scalar curvature turns out to be

$$R = \frac{3}{r^2} \left\{ \frac{2}{3}(1-\beta^2) + \beta^2 \frac{(K - D^2)r^4 + 1}{1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}} \right\} . \tag{2.14}$$

3 Examples of Bertrand spaces

In this section we identify within the common framework of the Bertrand spaces (2.1) and (2.2) several relevant specific cases that have been introduced in the literature by using rather different approaches. In particular, we show that the three classical Riemannian spaces of constant curvature [12], a generalization of the Darboux surface of type III [13] and the spaces of non-constant curvature studied by Iwai–Katayama [14, 15] belong to the family of Bertrand spaces.

3.1 Spaces with constant curvature

So far we have explicitly shown that Bertrand spaces are generically of non-constant curvature. Nevertheless the three classical Riemannian spaces with constant sectional curvature κ, the spherical ($\kappa > 0$), Euclidean ($\kappa = 0$), and hyperbolic spaces ($\kappa < 0$), can be recovered as special instances of Bertrand spaces. The metric of such spaces of constant curvature can be collectively written in terms of Poincaré coordinates $\mathbf{q} \in \mathbb{R}^3$

(coming from a stereographic projection in $\mathbb{R}^4$ [12]) or in geodesic polar coordinates $(\hat{r}, \vartheta, \varphi)$ as [27, 28]

$$\mathrm{d}s^2 = \frac{4}{(1+\kappa \mathbf{q}^2)^2}\,\mathrm{d}\mathbf{q}^2 = \mathrm{d}\hat{r}^2 + \frac{\sin^2(\sqrt{\kappa}\,\hat{r})}{\kappa}\,\mathrm{d}\Omega^2. \tag{3.1}$$

We remark that $\hat{r}$ is the distance along a minimal geodesic that joins the particle and the origin in our Riemannian 3-space; this geodesic distance does not coincide neither with $\tilde{r} = |\mathbf{q}|$ nor with r (2.4). By taking into account that the conformal function of the metric (3.1) is $f(\tilde{r}) = 2/(1+\kappa\tilde{r}^2)$, and by defining

$$\tilde{r} = \frac{1}{\sqrt{\kappa}} \tan\left(\sqrt{\kappa}\,\frac{\hat{r}}{2}\right) \quad \text{or} \quad r = \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}\ \hat{r}\right) \tag{3.2}$$

we find that the generic metric (2.3) reduces to (3.1).

This result directly comes from Bertrand spaces (2.1) and (2.2) by setting [6]:

$$\text{Type I}: \ \beta = 1 \quad K = -\kappa \qquad \text{Type II}_+: \ \beta = 2 \quad K = 0 \quad D = \kappa \tag{3.3}$$

which yields

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2 \mathrm{d}\Omega^2 \tag{3.4}$$

that is, $g(r) = (1-\kappa r^2)^{-1/2}$. By applying the change of radial coordinates (3.2), we obtain that the latter metric coincides with (3.1). Note also that the scalar curvature (2.10) is constant and equal to 6κ.

3.2 Darboux space of type III

The so-called 2D *Darboux spaces* were studied by Koenigs [29] and these are the only surfaces with non-constant curvature admitting two functionally independent integrals, quadratic in the momenta, that commute with the geodesic flow [29, 30, 31, 32, 33]. There are four types of such spaces. We shall focus on the Darboux surface of type III, $\mathcal{D}_{\rm III}$, whose metric in terms of isothermal coordinates [34] (u, v) is given by

$$\mathrm{d}s^2 = \mathrm{e}^{-2u}(1+\mathrm{e}^u)\,(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{3.5}$$

If we define new coordinates (q_1, q_2) through

$$q_1 = \mathrm{e}^{-u/2}\cos(v/2) \qquad q_2 = \mathrm{e}^{-u/2}\sin(v/2) \tag{3.6}$$

the metric (3.5) is transformed, up to a constant factor, into

$$\mathrm{d}s^2 = (1 + q_1^2 + q_2^2)(\mathrm{d}q_1^2 + \mathrm{d}q_2^2). \tag{3.7}$$

From this expression an ND generalization of $\mathcal{D}_{\rm III}$ was recently proposed in [13]. Here we consider the 3D version given by

$$\mathrm{d}s^2 = (k + \mathbf{q}^2)\mathrm{d}\mathbf{q}^2 \tag{3.8}$$

where k is an arbitrary real constant; this metric is clearly of the form (2.3) with

$$f(\tilde{r}) = \sqrt{k + \tilde{r}^2} \qquad r = F(\rho) = \mathrm{e}^{\rho}\sqrt{k + \mathrm{e}^{2\rho}}. \tag{3.9}$$

Hence

$$\rho = \frac{1}{2}\ln\left(\frac{-k \pm \sqrt{k^2 + 4r^2}}{2}\right) \tag{3.10}$$

which gives

$$g(r) = r\,\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{\sqrt{k^2 + 4r^2} \pm k}{2\sqrt{k^2 + 4r^2}} \tag{3.11}$$

so that the metric (3.8) is transformed into

$$\mathrm{d}s^2 = \frac{k^2 + 2r^2 \pm k\sqrt{k^2 + 4r^2}}{2(k^2 + 4r^2)}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2. \tag{3.12}$$

In this way we show that $\mathcal{D}_{\mathrm{III}}$ is in fact a Bertrand space of type $\mathrm{II}_{\pm}$ (2.2) provided that the Perlick parameters are taken as

$$\beta = 2 \qquad K = D^2 \qquad D = -2/k^2. \tag{3.13}$$

The corresponding scalar curvature reads

$$R = -6\,\frac{2k + \tilde{r}^2}{(k + \tilde{r}^2)^3} = -6\,\frac{2k + \mathrm{e}^{2\rho}}{(k + \mathrm{e}^{2\rho})^3} = 3\,\frac{k^4 + 2k^2r^2 - 2r^4 \mp k^3\sqrt{k^2 + 4r^2}}{r^6}. \tag{3.14}$$

3.3 Iwai–Katayama spaces

These are the spaces underlying the so-called 'multifold Kepler' systems introduced by these authors in [15]. Their metric is given by

$$\mathrm{d}s^2 = \tilde{r}^{\frac{1}{\nu}-2}(a + b\,\tilde{r}^{1/\nu})\left(\mathrm{d}\tilde{r}^2 + \tilde{r}^2\mathrm{d}\Omega^2\right) \tag{3.15}$$

where a and b are two real constants, while ν is a rational number. This metric is of the form (2.3) provided that

$$f(\tilde{r}) = \tilde{r}^{\frac{1}{2\nu}-1}(a + b\,\tilde{r}^{1/\nu})^{1/2} \qquad r = \; F(\rho) = \left(a\exp(\rho/\nu) + b\exp(2\rho/\nu)\right)^{1/2} \tag{3.16}$$

so that

$$\rho = \nu\ln\left(\frac{-a \pm \sqrt{a^2 + 4br^2}}{2b}\right) \tag{3.17}$$

with $b \neq 0$. From it we compute the Perlick function $g(r)$:

$$g(r) = r\,\frac{\mathrm{d}\rho}{\mathrm{d}r} = \nu\,\frac{\sqrt{a^2 + 4br^2} \pm a}{\sqrt{a^2 + 4br^2}}. \tag{3.18}$$

Therefore the Iwai–Katayama metric (3.15) can be written as

$$
\mathrm{d}s^2 = 2\nu^2\,\frac{a^2+2br^2 \pm a\sqrt{a^2+4br^2}}{a^2+4br^2}\,\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \tag{3.19}
$$

which is proven to belong to the Bertrand family $\mathrm{II}_{\pm}$ (2.2) under the identifications

$$
\beta^2 = \frac{1}{\nu^2} \qquad K = D^2 \qquad D = -\frac{2b}{a^2}. \tag{3.20}
$$

Hence, the 'multifold Kepler' metric is actually a subcase of the Perlick metric. The scalar curvature for these spaces turns out to be

$$
\begin{aligned}
R &= \frac{8ab(\nu^2-1)+4b^2(\nu^2-1)\tilde{r}^{1/\nu}+a^2(4\nu^2-1)\tilde{r}^{-1/\nu}}{2\nu^2(a+b\tilde{r}^{1/\nu})^3}\\
&= \frac{3a^4+6a^2br^2+8b^2(\nu^2-1)r^4 \mp 3a^3\sqrt{a^2+4br^2}}{4\nu^2b^2r^6}
\end{aligned} \tag{3.21}
$$

which acquires a simpler expression once the parameter D (3.20) is plugged in:

$$
R = \frac{2(\nu^2-1)}{\nu^2r^2} + 3\,\frac{1-Dr^2 \mp \sqrt{1-2Dr^2}}{\nu^2D^2r^6}. \tag{3.22}
$$

We stress that the Darboux space of type III with metric (3.12) is a particular case of these Iwai–Katayama spaces as it can be recovered by setting $b=1$, $a=k$ and $\nu=1/2$. Nevertheless a separate study of $\mathcal{D}_{\mathrm{III}}$ allows us to elaborate on the connections with Darboux surfaces and gives rise to a neater analysis of the relationships with KC and harmonic oscillator potentials.

4 KC and oscillator potentials on Bertrand spaces

It is well known that the KC potential $1/r$ ($r^2=\mathbf{q}^2=q_1^2+q_2^2+q_3^2$) in $\mathbb{R}^3$ is simply a multiple of the (minimal) Green function of the Laplacian $\Delta_{\mathbb{R}^3} := \sum_i \frac{\partial^2}{\partial q_i^2}$, whereas the harmonic oscillator potential is the inverse of its square. We shall extend this prescription, holding on the flat Euclidean space, to a generic curved manifold M with metric (2.3). As we shall see, such a prescription is fully consistent with all results previously known in the literature.

The Laplace–Beltrami operator on (M,g) can be written as

$$
\Delta_M = \frac{1}{\sqrt{g}}\sum_{i,j=1}^{3}\frac{\partial}{\partial q_i}\sqrt{g}\,g^{ij}\frac{\partial}{\partial q_j} \tag{4.1}
$$

in terms of an arbitrary set of local coordinates $\mathbf{q}=(q_1,q_2,q_3)$. In this formula, as usual, g^{ij} denotes the inverse of the matrix tensor g_{ij} in these coordinates and g is the determinant of g_{ij}. Let us define the radial function $U(|\mathbf{q}|)$, which can be expressed

in terms on any of the radial coordinates $\tilde{r}$, ρ or r given in (2.4), as the positive non-constant solution to the equation

$$\Delta_M U = 0 \quad \text{on} \quad M\backslash\{\mathbf{0}\}. \tag{4.2}$$

A short calculation using (2.3) and (2.4) shows that

$$\begin{aligned}\Delta_M U &= \frac{1}{\tilde{r}^2 f(\tilde{r})^3}\frac{\mathrm{d}}{\mathrm{d}\tilde{r}}\left(\tilde{r}^2 f(\tilde{r})\frac{\mathrm{d}U(\tilde{r})}{\mathrm{d}\tilde{r}}\right) = \frac{1}{F(\rho)^3}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(F(\rho)\frac{\mathrm{d}U(\rho)}{\mathrm{d}\rho}\right)\\ &= \frac{1}{r^2 g(r)}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{g(r)}\frac{\mathrm{d}U(r)}{\mathrm{d}r}\right).\end{aligned} \tag{4.3}$$

This means that the symmetric Green function U of the Laplace–Beltrami operator in M (see [35, 36] and references therein) is given by

$$U = \int^{\tilde{r}} \frac{\mathrm{d}\tilde{r}}{\tilde{r}^2 f(\tilde{r})} = \int^{\rho} \frac{\mathrm{d}\rho}{F(\rho)} = \int^{r} \frac{g(r)}{r^2}\,\mathrm{d}r \tag{4.4}$$

up to inessential additive and multiplicative constants. Its potential-theoretic interpretation now leads to define the *intrinsic KC potential* on the 3D manifold M as

$$\mathcal{U}_{\mathrm{KC}} := \alpha\, U \tag{4.5}$$

where α is an arbitrary constant. The *intrinsic harmonic oscillator potential* in M is defined to be proportional to the inverse square of the KC potential:

$$\mathcal{U}_{\mathrm{ho}} := \frac{\alpha}{U^2}\,. \tag{4.6}$$

In what follows we apply these results to the Perlick metrics (2.1) and (2.2).

4.1 Type I: intrinsic KC potential

By considering (2.9) and (4.4) we obtain the Green function U corresponding to the Bertrand space of type I with metric (2.1):

$$U = -\frac{1}{2\beta}\left(\tilde{r}^{-\beta} + K\tilde{r}^{\beta}\right) = -\frac{1}{2\beta}\left(\mathrm{e}^{-\beta\rho} + K\mathrm{e}^{\beta\rho}\right) = -\frac{1}{\beta}\sqrt{r^{-2}+K}. \tag{4.7}$$

Then the intrinsic KC and harmonic oscillator potentials on these spaces are defined by

$$\mathcal{U}_{\mathrm{KC}} = -\frac{\alpha}{2\beta}\left(\tilde{r}^{-\beta} + K\tilde{r}^{\beta}\right) = -\frac{\alpha}{2\beta}\left(\mathrm{e}^{-\beta\rho} + K\mathrm{e}^{\beta\rho}\right) = -\frac{\alpha}{\beta}\sqrt{r^{-2}+K} \tag{4.8}$$

$$\mathcal{U}_{\mathrm{ho}} = \alpha\,\frac{4\beta^2}{\left(\tilde{r}^{-\beta} + K\tilde{r}^{\beta}\right)^2} = \alpha\,\frac{4\beta^2}{\left(\mathrm{e}^{-\beta\rho} + K\mathrm{e}^{\beta\rho}\right)^2} = \alpha\,\frac{\beta^2}{r^{-2}+K}. \tag{4.9}$$

Next if we compare these expressions with the warping function appearing in the $(3+1)$D Bertrand spacetime (1.3),

$$V(r) = G + \sqrt{r^{-2}+K} \tag{4.10}$$

we find that this is exactly the *intrinsic KC potential* on the Bertrand space (2.1), $V = \mathcal{U}_{\mathrm{KC}} + G$, with $\alpha = -\beta$ and G playing the role of an additive constant of the potential.

4.2 Type II$_\pm$: intrinsic harmonic oscillator potential

In this case, the Green function $U(r)$ has exactly the expression (2.13). Hence the intrinsic KC and harmonic oscillator potentials on the curved spaces of type II$_\pm$ are given (only in terms of r) by

$$\mathcal{U}_{\rm KC}(r) = \mp\alpha\frac{\sqrt{2}}{\beta r}\left(1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}\right)^{1/2} \tag{4.11}$$

$$\mathcal{U}_{\rm ho}(r) = \frac{\alpha\beta^2 r^2}{2\left(1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}\right)}. \tag{4.12}$$

By taking into account the equations (1.4) and (4.12) we conclude that the underlying potential of the $(3+1)$D Bertrand spacetimes of type II$_\pm$, namely

$$V(r) = G \mp r^2\left(1 - Dr^2 \pm \sqrt{(1-Dr^2)^2 - Kr^4}\right)^{-1} \tag{4.13}$$

is exactly the *intrinsic harmonic oscillator* on the corresponding 3D Bertrand space (2.2), $V = \mathcal{U}_{\rm ho} + G$, provided that $\alpha = \mp 2/\beta^2$ and G being again an additive constant of the potential.

At this point some comments concerning the above results and those obtained by Perlick [6] seem to be pertinent. So far we have obtained all the functions $V(r)$ in the families I and II$_\pm$ of $(3+1)$D Bertrand spacetimes as, respectively, the intrinsic KC and oscillator potentials on the corresponding 3D Bertrand spaces. This identification holds for any value of β, while in the exhaustive analysis performed by Perlick it is shown that relativistic analogues of the KC and oscillator systems can only be obtained by considering physical constraints that lead to fixed values $\beta = 1$ and $\beta = 2$, respectively, of the Bertrand spacetimes (cf. propositions 4 and 5 in [6]). Note that, obviously, both results are not in contradiction since the definition of the Kepler/oscillator potentials on 3D Bertrand spaces is based on a different (non-relativistic) physical setting that does not exclude any value of β.

5 Examples of Bertrand spacetimes

In this section we firstly specialize the intrinsic KC and harmonic oscillator potentials for the particular Bertrand spaces described in section 3 and, secondly, we single out those Bertrand spacetimes associated to them.

5.1 Bertrand spacetimes from constant curvature spaces

Let us consider the classical Riemannian spaces of constant sectional curvature κ described in section 3.1 with metric (3.1) or (3.4). The Green function (4.4) is therefore given by

$$U = -\frac{1-\kappa\tilde{r}^2}{2\tilde{r}} = -\frac{\sqrt{1-\kappa r^2}}{r}. \tag{5.1}$$

The definition of the geodesic radial coordinate $\hat{r}$ (3.2) gives

$$\frac{\tan(\sqrt{\kappa}\,\hat{r})}{\sqrt{\kappa}} = \frac{r}{\sqrt{1-\kappa r^2}}. \tag{5.2}$$

In this way the corresponding KC and oscillator potentials are found to be

$$\text{Type I}: \qquad \mathcal{U}_{\rm KC} = -\alpha\,\frac{1-\kappa\tilde{r}^2}{2\tilde{r}} = -\alpha\sqrt{r^{-2}-\kappa} = -\alpha\,\frac{\sqrt{\kappa}}{\tan(\sqrt{\kappa}\,\hat{r})} \tag{5.3}$$

$$\text{Type II}_+: \qquad \mathcal{U}_{\rm ho} = \frac{4\alpha\tilde{r}^2}{(1-\kappa\tilde{r}^2)^2} = \frac{\alpha r^2}{1-\kappa r^2} = \alpha\,\frac{\tan^2(\sqrt{\kappa}\,\hat{r})}{\kappa} \tag{5.4}$$

which are, in this order, particular cases of the Bertrand potential of type I (4.8) and of the type II_+ (4.12) provided that the relations (3.3) have been introduced. Obviously we can always add the additive constant G. Notice that, within this framework, we have recovered the well known expressions for these potentials (see [27, 28] and references therein).

Consequently both the KC and harmonic oscillator potentials on these spaces of constant curvature lead to particular cases of Bertrand spacetimes; these are

$$\text{Type I}: \qquad {\rm d}s^2 = -\frac{{\rm d}t^2}{G+\sqrt{r^{-2}-\kappa}} + \frac{{\rm d}r^2}{1-\kappa r^2} + r^2{\rm d}\Omega^2 \tag{5.5}$$

$$\text{Type II}_+: \qquad {\rm d}s^2 = -\frac{{\rm d}t^2}{G - r^2\left(2(1-\kappa r^2)\right)^{-1}} + \frac{{\rm d}r^2}{1-\kappa r^2} + r^2{\rm d}\Omega^2 \tag{5.6}$$

where we have fixed $\alpha=-1$ and $\alpha=-1/2$, respectively.

5.2 A Bertrand–Darboux spacetime

In this case the Green function (4.4) of the metric (3.8) or (3.12) turns out to be

$$U = -\frac{\sqrt{k+\tilde{r}^2}}{k\tilde{r}} = -\frac{\sqrt{1+k\,{\rm e}^{-2\rho}}}{k} = -\frac{k\pm\sqrt{k^2+4r^2}}{2kr}. \tag{5.7}$$

By using this Green function we can define the intrinsic KC and oscillator potentials on $\mathcal{D}_{\rm III}$. However since such a space appears as a particular Bertrand space of type $\mathrm{II}_\pm$, *only* the oscillator potential can be considered in this context; namely

$$\text{Type II}_\pm: \quad \mathcal{U}_{\rm ho} = \frac{\alpha k^2\tilde{r}^2}{k+\tilde{r}^2} = \frac{\alpha k^2}{1+k\,{\rm e}^{-2\rho}} = \frac{2\alpha k^2 r^2}{k^2+2r^2\pm k\sqrt{k^2+4r^2}} \tag{5.8}$$

which is reproduced from (4.12) through the identifications (3.13). We recall that this was exactly the oscillator potential on $\mathcal{D}_{\rm III}$ worked out in [13] in terms of the radial coordinate $\tilde{r}=|\mathbf{q}|$. Thus its corresponding Bertrand spacetime of type $\mathrm{II}_\pm$ reads

$${\rm d}s^2 = -\frac{{\rm d}t^2}{G\mp k^2r^2\left(k^2+2r^2\pm k\sqrt{k^2+4r^2}\right)^{-1}} + \frac{k^2+2r^2\pm k\sqrt{k^2+4r^2}}{2(k^2+4r^2)}\,{\rm d}r^2 + r^2{\rm d}\Omega^2 \tag{5.9}$$

provided that $\alpha=\mp1/2$.

5.3 Bertrand spacetimes from Iwai–Katayama spaces

The Iwai–Katayama metric (3.15) or (3.19) gives rise to the following Green function

$$\begin{aligned} U &= -\frac{2\nu}{a}\sqrt{a\tilde{r}^{-1/\nu}+b} = -\frac{2\nu}{a}\sqrt{a\mathrm{e}^{-\rho/\nu}+b} \\ &= -\frac{\sqrt{2}\nu}{ar}\left(a^2+2br^2 \pm a\sqrt{a^2+4br^2}\right)^{1/2} \end{aligned} \tag{5.10}$$

which allows for the definition of the corresponding intrinsic KC and oscillator potentials. The results of section 3.3 show that only the oscillator potential can be constructed on these spaces in order to obtain a Bertrand spacetime of type II$_\pm$ by means of the relations (3.20); explicitly

$$\begin{aligned} \text{Type II}_\pm: \qquad \mathcal{U}_{\rm ho} &= \frac{\alpha a^2}{4\nu^2(a\tilde{r}^{-1/\nu}+b)} = \frac{\alpha a^2}{4\nu^2(a\mathrm{e}^{-\rho/\nu}+b)} \\ &= \frac{\alpha a^2 r^2}{2\nu^2}\left(a^2+2br^2 \pm a\sqrt{a^2+4br^2}\right)^{-1}. \end{aligned} \tag{5.11}$$

These results can be analysed in relation with the so called 'multifold Kepler' potential $V_{\rm IK}$ on (3.15) introduced in [15] which is given by

$$V_{\rm IK}(\tilde{r}) = \frac{\tilde{r}^{2-\frac{1}{\nu}}}{a+b\tilde{r}^{\frac{1}{\nu}}}\left(\mu^2\tilde{r}^{-2} + \mu^2 c\,\tilde{r}^{\frac{1}{\nu}-2} + \mu^2 d\,\tilde{r}^{\frac{2}{\nu}-2}\right) \tag{5.12}$$

where μ, c and d are real constants. We remark that this potential is of physical relevance as for $\nu = 1$ (together with some specific values of the parameters a, b, c and d as pointed out in [15]) it comprises the MIC–Kepler problem [16, 17, 18, 37, 38] as well as the Taub-NUT one [20, 21, 25, 39, 40, 41].

Now notice that the angular momentum of the system $\mathbf{L}^2$ conjugate to $\mathrm{d}\Omega^2$, appearing in the metric (3.15), is a constant of the motion (see e.g. [13]) so that the term $\mu^2\tilde{r}^{-2}$ can be reabsorbed within the kinetic energy term since $\mathbf{L}^2+\mu^2$ is obviously an integral of motion, μ^2 being an additive constant. Then the potential reduces, up to the multiplicative constant μ^2, to

$$V_{\rm IK}(\tilde{r}) = \frac{c+d\,\tilde{r}^{\frac{1}{\nu}}}{a+b\tilde{r}^{\frac{1}{\nu}}}. \tag{5.13}$$

This, in turn, means that $V_{\rm IK}$ can be regarded as a Möbius map on the positive real line with variable $x = \tilde{r}^{1/\nu} = \exp(\rho/\nu) \in \mathbb{R}^+$ under the action of the group $SL_2(\mathbb{R})$:

$$x \to V_{\rm IK} = g\cdot x = \frac{dx+c}{bx+a} \quad \text{where} \quad g = \begin{pmatrix} d & c \\ b & a \end{pmatrix} \in SL_2(\mathbb{R}).$$

Next if we add a constant G to the oscillator potential (5.11) we find that

$$\mathcal{U}_{\rm ho}(\tilde{r}) + G = \frac{c+d\,\tilde{r}^{\frac{1}{\nu}}}{a+b\tilde{r}^{\frac{1}{\nu}}} \tag{5.14}$$

provided that

$$G = c/a \qquad ad - bc = \frac{\alpha a^3}{4\nu^2}. \tag{5.15}$$

Note that the second relation is a constraint between the parameters of the potential and those of the metric, which arises from the requirement that the Möbius transformation be non-degenerate.

Consequently the 'multifold Kepler' potential $V_{\rm IK}$ (5.13) is actually an intrinsic harmonic oscillator (with constant d plus an additive constant c) on the Iwai–Katayama spaces studied in section 3.3.

Finally, we write the resulting metric of the associated Bertrand spacetimes (which are of type $\mathrm{II}_\pm$):

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\mathrm{d}t^2}{G \mp a^2 r^2 \left(a^2 + 2br^2 \pm a\sqrt{a^2+4br^2}\right)^{-1}} \\ &+ 2\nu^2 \, \frac{a^2 + 2br^2 \pm a\sqrt{a^2+4br^2}}{a^2+4br^2}\, \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2 \end{aligned} \tag{5.16}$$

where we have set $\alpha = \mp 2\nu^2$.

6 Concluding remarks

In this paper we have explicitly shown that the families of (3+1)D Bertrand spacetimes I and $\mathrm{II}_\pm$ can be neatly understood in terms of the 'intrinsic' KC and harmonic oscillator potential on the corresponding 3D Bertrand spaces. In this sense, Bertrand spaces can be simply described as those manifolds where the classical Bertrand's theorem still holds. Moreover, the resulting central potentials cannot be but the KC or the harmonic oscillator potentials. Our general description of Bertrand spaces and potentials has been also applied to some relevant cases which so far appeared in the literature as stemming from different approaches. In particular we have described in a unified way the intrinsic oscillator potential on the Darboux III and Iwai–Katayama spaces. As a byproduct of these examples we have proven that the so called 'multifold Kepler' potential, and therefore the MIC–Kepler and Taub-NUT problems, comes out in a natural way within a certain class of $(3+1)$D Bertrand spacetimes of type $\mathrm{II}_\pm$, thus suggesting that such potential should be actually regarded as an intrinsic harmonic oscillator.

It is worth recalling that the main algebraic feature of the KC and oscillator potentials on the ND Euclidean space is the fact that both Hamiltonian systems are *maximally superintegrable*: this means that both systems admit a maximum number $(2\,N-2)$ of functionally independent and globally defined constants of the motion that Poisson-commute with the Hamiltonian. Moreover, it is well known that all bounded orbits of any maximally superintegrable Hamiltonian system are periodic. Thus, one could conjecture that the KC and oscillator systems on generic Bertrand spaces should be maximally superintegrable. In order to prove this statement, the explicit form of the $2\cdot 3 - 2 = 4$ independent integrals of the motion that would commute with these

Hamiltonian flows should be found. Note that this result is already known in the literature for the 'ν-multifold Kepler systems' with any rational value of $\nu = 1/\beta$ [15]. We point out that, again, this problem is quite different to the one consisting in the study of the maximal superintegrability of the geodesic motion on the $(3+1)$D Bertrand spacetimes which is, in fact, solved (see proposition 6 in [6]) and imposes that $\beta = 1, 2$.

We will report on this question elsewhere [42]. For completeness, let us mention that this fits naturally into the framework of Hamiltonian systems endowed with an $sl(2)$ Poisson coalgebra symmetry [43, 44, 45]. It turns out that this coalgebra symmetry provides the *quasi-maximal superintegrability* of any spherically symmetric Hamiltonian system defined on a Bertrand spacetime (*i.e.* three of the above integrals). Therefore, in order to achieve the maximal superintegrability of the KC and oscillator potentials only one additional integral has to be found by direct methods. The maximal superintegrability of motion in the particular case of Iwai–Katayama spaces has already been established [39, 40, 41], and in fact its (complexified) symmetry algebra is isomorphic to that of the KC or harmonic oscillator potentials. Incidentally, it is worth stressing that this problem is related to similar questions studied by Koenigs in connection with the maximal superintegrability of geodesic flows on surfaces [13, 29, 30, 31, 32, 33].

As a final remark, we would like to mention that despite having focused our discussion on the $(3+1)$D case, the results here presented can be easily generalized to $(N+1)$ dimensions by rewriting the spherical symmetry in terms of an underlying $sl(2)$ Poisson coalgebra invariance.

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